Journal of Statistical Physics, Vol. 87, Nos. 3/4, 1997

Bootstrap Percolation in a Polluted Environment

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Received April 9, 1996; final December 17, 1996

Let a low density p of sites on the lattice $\mathbf{Z}^2$ be occupied, remove a proportion q of them, and call the remaining sites empty. Then update this configuration in discrete time by iteration of the following synchronous rule: an empty site becomes occupied by contact with at least two occupied nearest neighbors, while occupied and removed sites never change their states. If q/p^2 is large most sites remain unoccupied forever, while if q/p^2 is small, this dynamics eventually makes most sites occupied. This demonstrates how sensitive the usual bootstrap percolation rule (the $q=0$ case) is to the pollution of space.

KEY WORDS: Bootstrap percolation; phase transition.

1. INTRODUCTION

In this paper we consider a two-dimensional cellular automaton ξ_t which we call *bootstrap percolation in a polluted environment* (*BPPE*). Its state space is $\{0, 1, 2\}^{\mathbf{Z}^2}$, hence every site x can at time t be either 0 (empty), 1 (occupied), or 2 (removed). The very simple update rule is given as follows:

(BPPE1) If $\xi_t(x) > 0$, then $\xi_{t+1}(x) = \xi_t(x)$.

(BPPE2) If $\xi_t(x) = 0$ and there exist $y_1 \neq y_2$ so that $\|x - y_1\|_1 = \|x - y_2\|_1 = 1$ and $\xi_t(y_1) = \xi_t(y_2) = 1$, then $\xi_{t+1}(x) = 1$.

(BPPE3) Otherwise, $\xi_{t+1}(x) = 0$.

The initial state ξ_0 is a product measure specified by parameters p and q which measure the densities of 1's and 2's, i.e., $P(\xi_0(x) = 1) = p$,

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$P(\xi_0(x)=2)=q$, and $P(\xi_0(x)=0)=1-p-q$. Since every site changes state at most once, there exists a limiting configuration ξ_∞ which assigns to every site x its final state $\xi_\infty(x)$.

The standard bootstrap percolation rule is obtained by taking $q=0$. It has been extensively studied, together with various related models (see ref. 1 for a nice survey). One of the fundamental properties of this case is that there is no phase transition: $\xi_\infty \equiv 1$ for every $p>0$.[14-16] Many researchers have thus focused on studying the phase transition properties of *finite* systems,[2, 11, 12] on metastability issues of infinite systems,[2, 7] and on rates of convergence towards occupancy.[3, 4] One important result[2] is that the bootstrap percolation rule on a large $L \times L$ square experiences a phase transition from very sparse final occupancy to full final occupancy as $p \ln L$ changes from small to large. Our aim is to demonstrate that BPPE on the entire $\mathbf{Z}^2$ provides another natural context in which the bootstrap percolation dynamics experience a similar phase change, this time as q/p^2 changes from large to small. To make this more precise, we say that a set $S \subset \mathbf{Z}^2$ *percolates* if there exists an infinite self avoiding nearest-neighbor path contained in S. Then we define, for every fixed $q>0$,

$$p_c(q)=\sup\{p\colon \{\xi_\infty=1\} \text{ does not percolate}\}$$

We now state our main result.

Theorem 1.1. There exist finite positive constants c_1 and c_2 so that:

(1) If $q<c_1p^2$, then $P(\xi_\infty(x)=1)\to 1$ as $p\to 0$.

(2) If $q<c_2p^2$, then $P(\xi_\infty(x)=1)\to 0$ as $p\to 0$.

(3) $c_2^{-1/2} \leqslant \liminf_{q\to 0} q^{-1/2}p_c(q) \leqslant \limsup_{q\to 0} q^{-1/2}p_c(q) \leqslant c_1^{-1/2}$.

Theorem 1.1 quantifies how sensitive the bootstrap percolation rule is to the pollution of space by 2's. Assume that the density q of removed sites is fixed, and very low but positive, while p is the varying parameter. Then ξ_∞ is very far from total occupancy if p is relatively small; in fact ξ_∞ consists almost entirely of 0's if p is smaller than a constant times $\sqrt{q}$. On the other hand, the system hardly feels the pollution when p is larger than another constant times $\sqrt{q}$. We call the latter regime *supercritical* and the former *subcritical*. As the described phase transition happens when p is on the order of $\sqrt{q}$, it should be readily detectable even for small values of q, although appropriate simulations require huge arrays (of exponential size in $1/p$) due to rare nucleation.

We suspect that there exists a number c so that, for every $\varepsilon>0$, (1)–(3) in Theorem 1.1 hold with $c_1=c-\varepsilon$ and $c_2=c+\varepsilon$. Proving this

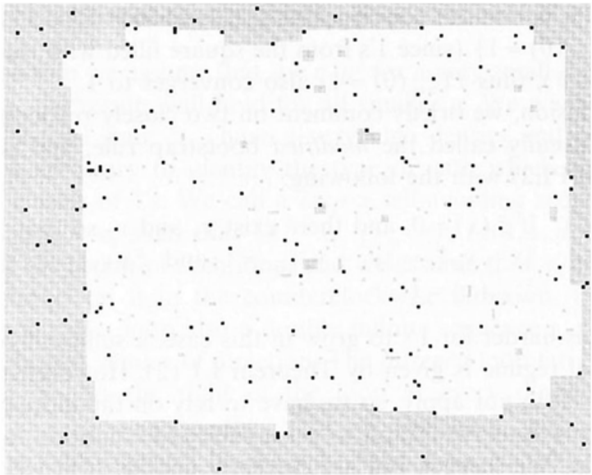

Fig. 1. A fixated state of the BPPE dynamics.

rigorously seems to be beyond our current techniques. Neither have we done enough statistical analysis of simulations to confidently conjecture a precise value of c. However, Fig. 1 does provide a "critical" picture of a 150×120 system with 1 boundary conditions (which are chosen to sidestep nucleation issues) (grey pixels represent color 1, while black pixels represent color 2). In the simulation, $q = 0.01$ was kept constant, and $p = 0.0238$ was the largest density for which the final percentage of occupied sites was below 1/2. While the size of this simulation is much too small to give a good estimate for c, it does illustrate how growth of 1's is stopped by 2's: the boundary of the final "frame" of 1's is what we later call a *blocking loop*. It is not hard to find a mean-field model for existence of long blocking loops [see (3.1)], which gives an idea about where the q/p^2 scaling comes from.

To provide an even more transparent explanation, we now give a simple argument which shows that, for any $\varepsilon > 0$, $q < p^{2+\varepsilon}$ implies that $P(\xi_\infty(x) = 1)$ converges to 1 as $p \to 0$. Let $N = \lfloor 1/p^{1+\varepsilon/3} \rfloor$, and call $x \in \mathbf{Z}^2$ *N-good* if the $(2N+1) \times (2N+1)$ square S_x centered at $(2N+1)\,x$ contains no 2 and at least one 1 in each of its rows and columns. Let $\mathscr{C}$ be the connected component of N-good sites which includes the origin and G the event that $\mathscr{C}$ includes a site x for which all sites in S_x are initially in the state 1. Then note that

$$P(x \text{ is not } N\text{-good}) \leqslant (2N+1)^2\, q + 2(2N+1)\, e^{-(2N+1)p} \to 0 \qquad \text{as} \quad p \to 0$$

Therefore $P(\mathscr{C}$ is infinite) converges to 1. However, $P(G) \geqslant P(\mathscr{C}$ is infinite) and $G \subset \{\xi_\infty, (0) = 1\}$ (since 1's from the square filled with them spread to S_x for all $x \in \mathscr{C}$), thus $P(\xi_\infty(0) = 1)$ also converges to 1.

In conclusion, we briefly comment on two closely related models. The first one is usually called the *modified* bootstrap rule, and is defined by replacing (BPPE2) with the following:

(BPPE2′) If $\xi_t(x) = 0$, and there exist y_1 and y_2 so that $\|x - y_1\|_1 = \|x - y_2\|_1 = \|y_1 - y_2\|_\infty = 1$ and $\xi_t(y_1) = \xi_t(y_2) = 1$, then $\xi_{t+1}(x) = 1$.

Since it is harder for 1's to grow in this case, a sufficient condition for the subcritical regime is given by Theorem 1.1 (2). However, the methods from Section 2 do not apply, so we have to rely on the more robust argument in the previous paragraph to get a sufficient condition for the supercritical regime. We suspect, but are currently unable to prove, that in this case the scaling differs from q/p^2 by a logarithmic correction.

Although BPPE is interesting in its own right, our interest in it was sparked by our previous work in competition of growth models.[8] A version of the *multitype threshold voter model (MTVM)* ξ'_t on state space $\{0, 1, 2\}^{\mathbf{Z}^2}$ can be defined by making the rules (MTVM1) and (MTVM3) identical to (BPPE1) and (BPPE3), respectively, and adding:

(MTVM2) If $\xi'_t(x) = 0$, and there exit a *unique* $k \in \{1, 2\}$ such that there are $y_1 \neq y_2$ with $\|x - y_1\|_1 = \|x - y_2\|_1 = 1$ and $\xi'_t(y_1) = \xi'_t(y_2) = k$, then $\xi'_{t+1}(x) = k$.

The two nonzero colors of MTVM hence grow over 0's using the bootstrap percolation rule and their competition leads to a standoff wherever they meet. Assume that the 1's and 2's start off equally matched: the initial state is a product measure with $P(\xi'_0(x) = 1) = P(\xi'_0(x) = 2) = p$ for some small $p > 0$. In refs. 7 and 8 we studied *supercritical* growth models extensively; we showed that they have no trouble overcoming a low density of removed sites, and that competition between them results in a tessellation of the available space. By contrast, the growth model studied here is *critical*; although it shares some features of the supercritical ones, namely rare nucleation and asymptotic shape,[9, 10] the results from ref. 8 fail to hold as testified by Theorem 1.1 and its consequences, one of which is stated below.

Corollary 1.2. As $p \to 0$, $P(\xi'_\infty(x) \neq 0) \to 0$.

The rest of this paper contains the proof of Theorem 1.1. In Section 2 we prove part (1) and the upper bound in (3), while Section 3 establishes the remaining bounds.

2. THE SUPERCRITICAL REGIME

In this section we assume that $q = c_1 p^2$ for a very small c_1. By obvious monotonicity, the results will hold for all smaller q. We want to establish that 1's in the final state ξ_∞, have a very high density and percolate. To this end, it is necessary to identify the type of path which can block the growth of a cluster of 1's. We call a *loop* a self-avoiding sequence of sites $l: x_0, x_1, x_2, ..., x_n = x_0$ such that $\|x_i - x_{i-1}\|_\infty = 1$, $i = 1, ..., n$. Naturally, n will be called the *length* of such loop, and we assume that x_i are numbered so that we travel on it in the counterclockwise direction. For technical reasons we will call a loop also a doubly infinite sequence $..., x_{-1}, x_0, x_1, ...$ with some arbitrary choice of direction. The *fattened* loop fat(l) is obtained by adding a site to the right of the loop at every diagonal move: if $\|x_{x+1} - x_i\|_1 = 2$, add y to the right of the loop with $\|y - x_i\|_1 = \|y - x_{i+1}\|_1 = 1$ and expand the loop to $..., x_i, y, x_{i+1}, ...$. A fattened loop hence ends up making no diagonal moves, but may no longer be self-avoiding. A *thick loop* is a sequence of sites $l: x_0, x_1, x_2, ..., x_n = x_0$ such that $\|x_i - x_{i-1}\|_1$, $i = 1, ..., n$, and is such that it makes no U-turns, i.e., $x_i \neq x_{i+2}$ for $i = 0, ..., n-2$. We also require that a thick loop never crosses itself, although it can retrace a part of itself *once* (that is, some sites can be visited twice).

A *left* (resp. *right*) *blocking loop* l' is a thick loop such that the following hold:

(BL1) All sites on l' which are 1 at time $t = 0$ are preceded by a 2 and succeeded by a 2.

(BL2) If l' makes a left turn (resp. a right turn) at x_i, i.e., $x_{i+1} - x_i$ is a 90-degree (resp. a -90-degree) rotation of $x_i - x_{i-1}$, then there is a site y such that $\xi_0(y) = 2$ and $\|x_i - y\|_\infty \leqslant 1$.

(BL3) All the 2's used in (BL2) are used only once; that is, there is a one-to-one assignment of described sites y to every left turn (resp. right turn).

We start with two geometric lemmas

Lemma 2.1. Assume that $\xi_0(x) = 1$, and that the connected component of x in $\{\xi_\infty = 1\}$ does not percolate. Then there exists a finite right blocking loop surrounding x.

Proof. Start by the loop l which consists entirely of 0's and 2's in ξ_∞, includes x in its interior, is minimal with respect to the set of sites it contains inside, and is also of minimal length. Then arbitrarily pick a first site on l and proceed to successively eliminate diagonal turns in the direction of the loop. In this process, we may encounter one of the 11 cases, depicted in Fig. 2.

The diagonal move to be eliminated is drawn, and all possible next moves. The symbol □ denotes sites on the loop l and the single arrows indicate the direction of l. The double arrows indicate the direction of

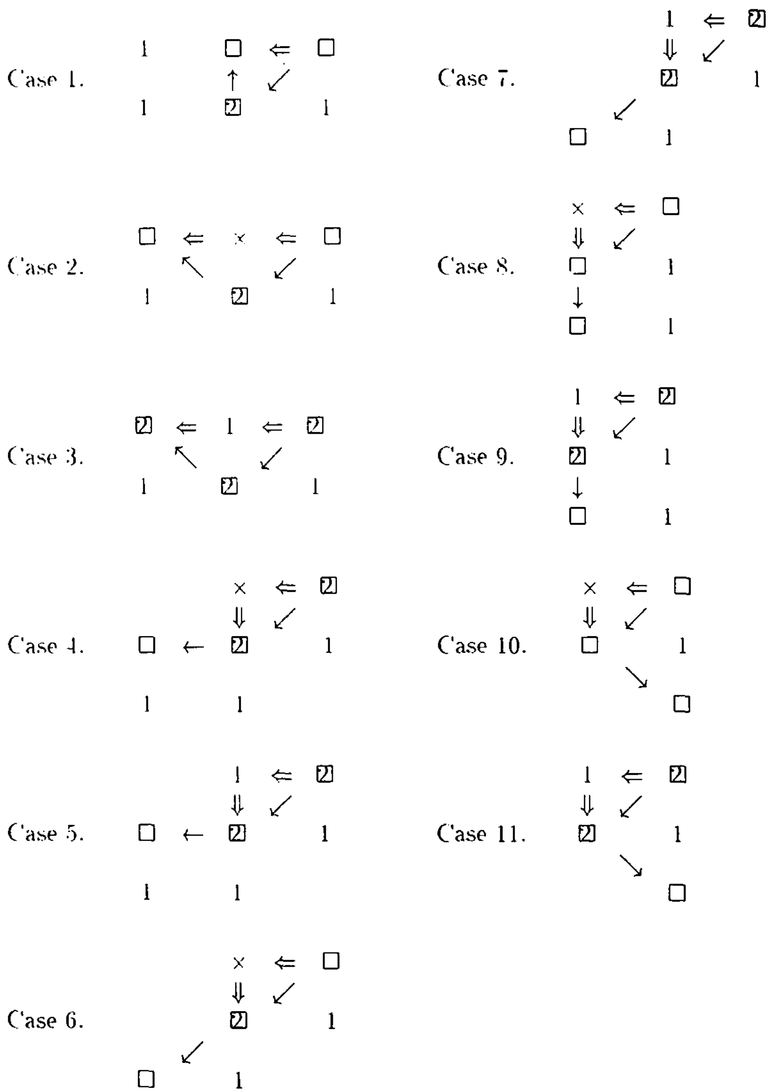

Fig. 2. Eliminating diagonal turns of l.

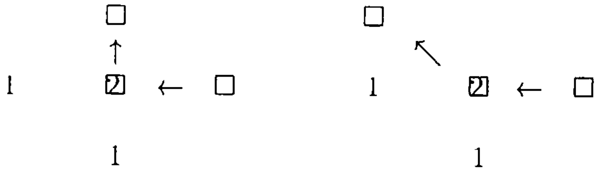

Fig. 3. Right turns of l.

fat(l). Those sites which are labeled by a number must be in a specified state in ξ_x, and $\times$ indicates that a site can either be a 0 or a 2. For example, in case 1, the three 1's are there because the number of sites inside l cannot be reduced, and this forces a 2 between the two 1's. (Note how this argument breaks down for the modified bootstrap rule in cases 1 and 2.) Right turns may be created by this procedure, but in every case there is a 2 sufficiently close to satisfy (BL2). Also, as we can see from cases 3, 5, 7, 9, and 11, (BL1) is satisfied.

Furthermore, in the case l itself makes a right turn, the local configuration must be as in Fig. 3.

The elimination of diagonal turns may result in creating a non-self-avoiding loop. Even more, a U-turn may be created in the three cases depicted in Fig. 4 (other cases are rotation or mirror images of these).

In these cases, we just throw away the "dangling end" in the loop. Again, a right turn may be created this way, but in all such cases again there is a 2 sufficiently close. Finally, (BL3) is clearly satisfied. ∎

We omit the similar proof of the next lemma.

Lemma 2.2. Assume that $\{\xi_x = 1\}$ percolates and that $\xi_x(x) = 0$. Then either there exists a finite left blocking loop around x or there exists an infinite blocking loop somewhere in $\mathbf{Z}^2$.

From now on, we denote by C the "generic constant," a positive number whose value is not important and changes from one appearance to another. We will also assume without loss of generality, that $1/p$ is an even integer.

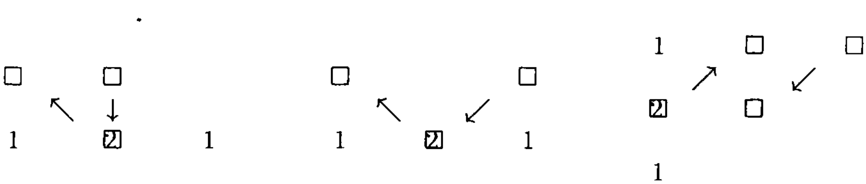

Fig. 4. Local configurations at U-turns.

Lemma 2.3. For all $N \geqslant p^{-3}$, P (there exists a right blocking loop surrounding 0 and contained outside $[-N, N]^2) \leqslant e^{-CpN}$.

Proof. Assume that a right blocking loop l' surrounding 0 and contained outside $[-N, N]^2$ exists. Now let r be the number of right turns this loop makes. Then it must make $r+4$ left turns (since it is a loop with winding number 1). We identify the turn by the site at which it is made. The cost of a right turn is at least $9q$. Now take a site on l' which is at $\|\cdot\|_\infty$ distance at least 2 from the set of right turns of l'. The cost of such a site together with the preceding and succeeding sites is at least $1-p+2q<1-p/2$. Of course, we have to take into the account the fact that any such site may be visited twice. Using the fact that, for any $\delta \geqslant 0$, $\sup_{0 \leqslant k \leqslant n} \binom{n}{k} \delta^k \leqslant e^{\delta n}$, we then get that the probability that such an l exists is bounded above by

$$\begin{aligned}
&\sum_{n \geqslant N} n \cdot \sum_{r=0}^{n} \binom{n}{r}\binom{n}{r+4} 9^r q^r (1-p/2)^{n/6-25r} \\
&\quad = \sum_{n \geqslant N} n \cdot \sum_{r=0}^{n} \binom{n}{r}\left(\frac{9q}{(1-p/2)^{25}}\right)^{r/2} \\
&\quad\quad \times \binom{n}{r+4}\left(\frac{9q}{(1-p/2)^{25}}\right)^{(r+4)/2} (9q)^{-2}(1-p/2)^{n/6+50} \\
&\quad \leqslant \sum_{n \geqslant N} Cn^2 q^{-2} e^{10\sqrt{qn}} e^{-pn/12} \\
&\quad \leqslant CN^2 p^{-5} e^{-(1-120\sqrt{c_1})PN/12}
\end{aligned}$$

which ends the proof. ∎

We omit the proof of next lemma, as it is an easy adaptation of the one above.

Lemma 2.4. With probability one there are no infinite blocking loops.

Lemma 2.5. The probability that there is a finite left blocking loop around the origin is bounded above by Cp.

Proof. Again, if such a loop exists, it must make r right turns and $r+4$ left turns for some $r \geqslant 0$. As in the proof of Lemma 2.3, we can bound the probability that such a loop exists by

$$\sum_{n \geqslant 4} n \cdot \sum_{r=0}^{n} \binom{n}{r} \binom{n}{r+4} 9^{r+4} q^{r+4} (1-p/2)^{n/6-25r-4}$$

$$\leqslant \sum_{n \geqslant 4} Cn^2 q^2 e^{(120\sqrt{q}-p)n/12}$$

$$\leqslant Cp^4 \sum_{n \geqslant 4} n^2 e^{-(1-120\sqrt{c_1})pn/12}$$

$$\leqslant Cp^4 \cdot p^{-3}$$

which ends the proof. ▮

Proof of (1) and the Upper Bound in (3) in Theorem 1.1. Let H be the event that $\xi_0(x) = 1$ for all $x \in [-p^{-3}, p^{-3}]^2$ and that there is no finite right blocking loop surrounding the origin. By Lemma 2.3,

$$P(H) > p^{(2p^{-3}+1)^2}(1-e^{-Cp^{-2}}) > 0$$

By Lemma 2.1, $H \subset \{\{\xi_\infty = 1\}$ percolates$\}$. Finally, by the ergodic theorem, $P(\xi_\infty = 1$ percolates$) = 1$.

Furthermore, by what we have proved so far, and Lemmas 2.2, 2.4, and 2.5, $P(\xi_\infty(x) = 0) \leqslant Cp$.

3. THE SUBCRITICAL REGIME

In this section, we assume that $q = c_2 p^2$ for a large c_2 and demonstrate that, for small enough p, 1's in ξ_∞ have a very low density and do not percolate. Hence we need to show that blocking loops of previous section are quite likely to exist. As is usual in problems of this type, we resort to a comparison with an oriented percolation model. Roughly, the idea is then to use the *left* blocking loops to prevent the 1's from reaching a typical site from the outside, and in addition we make these loops short enough so that the dynamics inside them is unlikely to add many 1's (in this part of the argument, we use methods from ref. 2).

To define the percolation model, fix a positive integer $a > 0$ (which at this point should be thought of as a parameter, although we later set $a = 200$). As usual, $B_\infty(x, r)$ will stand for the discrete $(2r+1) \times (2r+1)$ box centered at x. Call a sequence $x_0, x_1, x_2, ..., x_n$ an *NE-path* if the following conditions hold:

(NEP1) For every $i \geqslant 0$ either $x_{i+1} = x_i + e_1$ or $x_{i+1} = x_i + e_2$.

(NEP2) $x_{i+1} = x_i + e_2$ for $i = 0, ..., a$ and $x_{i+1} = x_i + e_1$ for $i = n-a, ..., n$.

(NEP3) If $x_{i+1}=x_i+e_1$ and $x_i=x_{i-1}+e_2$ ($i=1, 2,...$), then $\xi_0(x_i)=2$.
(NEP4) For every $i\geqslant 0$, $1\notin B_\infty(x_i, a)$.

Such path can be either finite or infinite. The existence of an infinite (or a very long) NE-path is equivalent to the survival of the following two-particle oriented percolation model: label particles u and r; a u gives birth to another u at the site immediately above, an r gives birth to two particles: a u immediately above and an r immediately to the right. In addition, a u also gives birth to an r immediately to the right if it happens to be on top of a 2, while both types of particles die as soon as they come within l^∞-distance a of a 1. A mean-field approximation of such model would be governed by the following linear system:

$$\begin{aligned} U' &= -(2a+1)^2 pU+R \\ R' &= qU-(2a+1)^2 pR \end{aligned} \tag{3.1}$$

In this system, U and R both converge to infinity as time progresses as soon as $q>(2a+1)^4 p^2$.

As usual in proving survival of interacting particle systems, an appropriate rescaling argument will be a decisive step in completing the proof of Theorem 1.1; this is the key point in the proof of our next lemma. In its statement, x_0 is the point $(\frac{1}{2}p^{-2}, 0)$, y_0 is the point $(p^{-11}, \frac{3}{4}p^{-11}-\frac{1}{2}p^{-2})$, S is the convex hull of the four points $(0, 0)$, $p^{-2}e_1$, $y_0+p^{-2}e_2$, and y_0, and its *interior* inter(S) are those points in S without any nearest neighbors outside S.

Lemma 3.1. The probability that there exist an *NE*-path connecting x_0 and y_0 which is, apart from x_0 and y_0, entirely included in inter(S) is at least Cp^2.

Proof. Fix a small $\alpha>0$ and let $L=\lfloor\alpha/p\rfloor$. We will call a site $x\in\mathbf{Z}^2$ a *rescaled open site* if:

(ROS) There exists a site $z\in[Lx+ae_2, Lx+(L-1-a)e_2]$ and a sequence of sites $x_0, x_1,..., x_n=z$ so that (NEP1), (NEP2), and (NEP3) hold, and $1\notin B_\infty(x_i, a)$ for $i=0,..., n-a$.

Claim. Fix an $\varepsilon>0$. Then α can be chosen small enough, and then c_2 large enough, so that the conditional probability P (both $x+e_1$ and $x+e_1+e_2$ are rescaled open sites $|x$ is a rescaled open site) $\geqslant 1-\varepsilon$.

To prove the Claim, let $z\in[Lx+ae_2, Lx+(L-1-a)e_2]$ be a site which satisfies (ROS). Notice that the condition does not involve sites with first coordinate larger than that of Lx. Therefore, we can make sure that

$x+e_1$ is a rescaled open site simply by demanding that there is no 1 in $\bigcup_{j=0}^{L}(B_\infty(z+je_1,a)\cap(Lx+[0,L]^2))$. This happens with probability at least $1-\varepsilon/4$ if α is small enough. Let $z_k=z+k(2a+1)e_1$, $k=1,\ldots,\lfloor L/(2a+1)\rfloor$, and let H_k be the event that there are no 1's in $[z_k,z_k+2Le_2]+B_\infty(0,a)$. Again, if α is small enough, $P(H_k)\geqslant 3/4$, and H_k are independent, so that the probability that more than $L/(3a)$ of them happen is at least $1-\varepsilon/4$ for small p. Let K be the random set of indices k such that H_k happens. Let H'_k be the event that there is a 2 in

$$[Lx+k(2a+1)e_1+(5L/4)e_2,\ Lx+k(2a+1)e_1+(7L/4)e_2]$$

and that H_k happens. If the cardinality of K is at least $L/(3a)$, then there are at least $L^2/(6a)$ sites where a 2 would make $\bigcup_{k\in K}H'_k$ happen. If c_2 is large enough, then, $P(\bigcup_{k\in K}H'_k)\geqslant 1-\varepsilon/4$. Finally, assume that k_0 is the smallest k so that H'_k happens, and assume that a site

$$y\in[Lx+k_0(2a+1)e_1+(5L/4)e_2,\ Lx+k_0(2a+1)e_1+(7L/4)e_2]$$

has $\xi_0(y)=2$. Then the probability that there is no 1 in

$$\bigcup_{j=0}^{L}(B_\infty(y+je_1,a)\cap(L(x+e_2)+[0,L]^2))$$

is again at least $1-\varepsilon/4$. This construction makes both $x+e_1$ and $x+e_1+e_2$ resealed open sites with probability at least $1-\varepsilon$ and thus proves the Claim above.

The Claim essentially says that resealed open sites form a 1-dependent oriented site percolation process in which sites are open with probability very close to 1. Now let S_r be the set of all sites x so that $Lx+[0,L-1]^2\subset S$. Moreover, let $x_{0r}=(\lfloor\frac{1}{2}p^{-2}/L\rfloor+1,0))$ and $y_{0r}=(\lfloor p^{-11}/L\rfloor-2,\lfloor(\frac{3}{4}p^{-11}-\frac{1}{2}p^{-2})/L\rfloor-1)$. Assume that x_{0r} is a rescaled open site. Let G be the event that x_{0r} is a rescaled open site and that there exists a sequence of rescaled open sites $x_{0r}=w_0,w_1,\ldots,w_m=y_{0r}\in S_r$ such that either $w_{i+1}=w_i+e_1$ or $w_{i+1}=w_i+e_1+e_2$ for every $i=0,\ldots,m-1$. A standard contour argument along the lines of ref. 5 (see also ref. 6) then establishes that, for a small enough ε, $P(G\,|\,x_{0r}$ is a rescaled open site$)\geqslant 0.5$. However, the probability that x_{0r} is a rescaled open site is at least $C\cdot P$ (at least one 2 in $x_0+[L/4,3L/4]\,e_2\geqslant Cp$). Finally, conditioned on G, the probability that y_0 is connected to x_0 by an NE-path described in the statement is again at least Cp. This ends the proof. ∎

Lemma 3.2. Let G_x be the event that set $B_\infty(x,p^{-11})\setminus B_\infty(x,\frac{1}{8}p^{-11})$ contains a left blocking loop l which includes no site in $\{\xi_0=1\}+B_\infty(0,a)$. Then $P(G_x)\geqslant 1-e^{-C/p}$.

Proof. We can assume that $x=0$. Take the convex hull of the following four points: $(\frac{1}{2}p^{-11}, 0)$, $(p^{-11}, 0)$, $(0, \frac{1}{4}p^{-11})$, $(0, \frac{1}{2}p^{-11})$, and add its reflections across the two axes and across the origin to obtain an annular region A. This region can be chopped into Cp^{-9} concentric annular regions of width p^{-2}, each of which (by Lemma 3.1) independently contains a left blocking loop with probability Cp^8. ∎

Take a set $A \subset \mathbf{Z}^2$ and fix a realization of ξ_0. Switch all the sites in A^c to 2 and all the 2's in A to 0. Then run the BPPE dynamics on such a configuration, and let $\phi(A)$ be the set all sites in A which ever become 1. As into ref. 2, a set A will be called *internally* spanned if $\phi(A)=A$.

Lemma 3.3. Let G'_x be the event that $x \notin \phi(B_\infty(x, p^{-11}))$ and that $\phi(B_\infty(x, p^{-11}))$ contain no site outside $\{\xi_0=1\}+B_\infty(0, 100)$. Then $P(G'_x) \geqslant 1-Cp$.

Proof. Again, assume that $x=0$. Let

$$H_1 = \{\text{there is an } y \in B_\infty(0, 200) \text{ with } \xi_0(y)=1\}$$

$$H_2 = \{\text{there exists an internally spanned rectangle inside } B_\infty(0, p^{-11}) \text{ with its longest side in } [50, 100]\}$$

In ref. 2 it is proved that $(G'_0)^c \subset H_1 \cup H_2$. However, a necessary condition for a rectangle to be internally spanned is that every second row and every second column must contain at least one 1, and therefore

$$P(H_2) \leqslant Cp^{-22} \cdot P(\text{at least 25 sites with color 1 on } 100^2 \text{ fixed sites}) \leqslant Cp^3$$

Moreover, $P(H_1) \leqslant Cp$, and these two estimates end the proof. ∎

Proof of (2) and the Lower Bound in (3) of Theorem 1.1. Let $a=200$. Then, if both G_x and G'_x happen, no site from the outside of the blocking loop guaranteed by G_x can influence the dynamics inside it, but G'_x ensures that the dynamics inside it does not occupy x. Hence $G_x \cap G'_x \subset \{\xi_\infty(x) \neq 1\}$, thus $P(\xi_\infty(x)=1) \leqslant Cp$, proving (2).

Let $M=\frac{1}{8}p^{-11}$ and this time call $x \in \mathbf{Z}^2$ a *rescaled closed site* if $G_{(2M+1)x} \cap G'_{(2M+1)x}$ happens. Then $P(x \text{ is a rescaled closed site}) \geqslant 1-Cp$ and any x, y with $\|x-y\|_\infty \geqslant 16$ are rescaled closed sites independently. Hence, for a small enough p, the complement of the set of resealed closed sites does not percolate. Therefore, if $\{\xi_\infty=1\}$ were to percolate, an infinite nearest neighbor path would have to cross $B_\infty(x, M)$ for some rescaled closed site x. But such a path would then have to cross the corresponding left blocking loop, which is impossible, since sites on that loop can never become 1. This contradiction ends the proof. ∎

ACKNOWLEDGMENTS

This problem was suggested to the second author by Roberto Schonmann, while it arose independently through the collaboration of the first author with David Griffeath. The authors wish to thank Roberto and David for many constructive discussions and helpful advice. During the development of this work, J. G. was partially supported by research grant J1-6157-0101-94 from Slovenia's Ministry of Science and Technology and E. M. was supported in part by U.S. NSF DMS 94-00644.

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